



6.2 Finding solutions to a system of linear equations





Introduction

- In this topic, we will
 - Describe the zero equation
 - Define homogeneous linear equations
 - Review the properties of linear equations
 - Consider all possibilities with a single equation in one or more unknowns
 - Consider issues with two linear equations but only one unknown
 - Solve all three cases with two linear equations in two unknowns:
 - A unique solution, no solution and infinitely many solutions
 - Look at higher-order systems of linear equations





Zero equations

- The *zero* equation in n unknowns is of the form:

$$0 x_1 + 0 x_2 + 0 x_3 + \cdots + 0 x_n = b_1$$

- If $b_1 = 0$, there are infinitely as every solution vector $\mathbf{x} =$
is valid solution
 - Every variable is a free variable

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

- However, if $b_1 \neq 0$, there are no solutions
 - No sum of zeros will ever equal a non-zero target value
- From this point on, we will assume a linear equation has at least one non-zero coefficient of at least one of the variables





Homogeneous linear equations

- Definition:
 - A linear equation that has a target value (or right-hand side) equal to zero is said to be a *homogenous linear equation*.
 - All other linear equations are *non-homogeneous*.
- For example,

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

$$a_1x_1 + a_2x_2 + a_3x_3 = 1$$





Homogeneous linear equations

Theorem

$\mathbf{x} = \mathbf{0}_n$ is a solution to a linear equation in n unknowns if and only if that linear equation is homogenous.

Proof:

Substituting in $x_1 = x_2 = x_3 = \cdots = x_n = 0$ into the linear equation $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$,

we get $a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 0 + \cdots + a_n \cdot 0 = 0 = b$

Thus, the equation is satisfied if and only if $0 = b$. ■





Properties of linear equations

Theorem

A vector $\mathbf{x}$ is a solution to a linear equation if and only if it is a solution to a non-zero scalar multiple of that linear equation.

Proof:

Given $\mathbf{x}$, assume $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$

$$c(a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n) = cb$$

$$(ca_1)x_1 + (ca_2)x_2 + (ca_3)x_3 + \cdots + (ca_n)x_n = cb$$

Thus $\mathbf{x}$ satisfies the equation multiplied by c .

Given $\mathbf{x}$, assume $(ca_1)x_1 + (ca_2)x_2 + (ca_3)x_3 + \cdots + (ca_n)x_n = cb$

As c is non-zero, multiply both sides by c^{-1}

$$c^{-1}((ca_1)x_1 + (ca_2)x_2 + (ca_3)x_3 + \cdots + (ca_n)x_n) = c^{-1}cb$$

Thus, $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$ so $\mathbf{x}$ satisfies the original equation. ■





Properties of linear equations

Theorem

If a vector $\mathbf{x}$ is a solution to two linear equations,
it is also a solution to the sum of those two linear equations.

Proof:

$$\text{Given } \mathbf{x}, \quad a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n = b_2$$

Adding these two yields:

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n + a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n = b_1 + b_2$$

Grouping by the x s:

$$(a_{1,1} + a_{2,1})x_1 + (a_{1,2} + a_{2,2})x_2 + (a_{1,3} + a_{2,3})x_3 + \cdots + (a_{1,n} + a_{2,n})x_n = b_1 + b_2$$

But this says $\mathbf{x}$ satisfies the sum of the two linear equations. ■





One equation

- Consider one equation and one unknown, assuming the coefficient $a_{1,1} \neq 0$,

$$a_{1,1}x_1 = b_1$$

- There is a unique solution

$$x_1 = \frac{b_1}{a_{1,1}}$$





One equation

- Consider one equation and $n > 1$ unknowns, assuming the coefficient $a_{1,1} \neq 0$,

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n = b_1$$

- There is a infinitely many solutions with the solution vector

$$\mathbf{x} = \begin{pmatrix} \frac{b_1 - a_{1,2}x_2 - a_{1,3}x_3 - \cdots - a_{1,n}x_n}{a_{1,1}} \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$





Two equations

- Suppose you had two equations, but only one unknown:

$$a_{1,1}x_1 = b_1$$

$$a_{2,1}x_1 = b_2$$

- The first equation says $x_1 = \frac{b_1}{a_{1,1}}$ and the second says $x_1 = \frac{b_2}{a_{2,1}}$
- If these are different, no solution exists

$$2x_1 = 5$$

$$4x_1 = 7$$

- If they are equal, $\frac{b_1}{a_{1,1}} = \frac{b_2}{a_{2,1}}$, this can only happen if Eqn 1 is a scalar multiple of Eqn 2

$$2x_1 = 5$$

$$-3x_1 = -7.5$$





Two equations

- Suppose you had these two linear equations with two unknown:

$$x_1 + 2x_2 = 1$$

$$3x_1 - 2x_2 = 9$$

- You now add -3 times Eqn 1 onto Eqn 2:

$$x_1 + 2x_2 = 1$$

$$-8x_2 = 6$$

- Thus, $x_2 = -0.75$
- Substituting this into Eqn 1, we have

$$x_1 + 2(-0.75) = 1$$

- Thus, $x_1 = 2.5$
- Thus, the solution vector is $\mathbf{x} = \begin{pmatrix} 2.5 \\ -0.75 \end{pmatrix}$ and this is the unique solution.





Equivalent systems of linear equations

- Given a system of linear equations, $\mathbf{x}$ is a solution to that system of linear equations if and only if $\mathbf{x}$ is also a solution to any of the following operations performed on that system of linear equations:
 1. Multiplying any one equation by a non-zero scalar.
 2. Adding a multiple of one equation onto another equation.
 3. Swapping two equations.
- If two systems of linear equations are equivalent, we will denote this with a tilde:

$$\begin{array}{rcl} x_1 + 2x_2 = 1 & \sim & x_1 + 2x_2 = 1 \\ 3x_1 - 2x_2 = 9 & & -8x_2 = 6 \end{array}$$

Adding $-3 \times \text{Eqn 1}$ onto Eqn 2.





Two equations

- Suppose you had these two linear equations with two unknown:

$$x_1 + 2x_2 = 1$$

$$-3x_1 - 6x_2 = 9$$

- You now add 3 times Eqn 1 onto Eqn 2:

$$x_1 + 2x_2 = 1$$

$$0 = 12$$

- This says $0x_1 + 0x_2 = 12$, which is impossible.
- Thus, no solution exists.





Two equations

- Suppose you had these two linear equations with two unknown:

$$x_1 + 2x_2 = 1$$

$$-3x_1 - 6x_2 = -3$$

- You now add 3 times Eqn 1 onto Eqn 2:

$$x_1 + 2x_2 = 1$$

$$0 = 0$$

- The last equation is the zero equation.
 - Thus, this equation no longer puts any constraints on any solution.
- Thus, the only equation remaining is the first,
thus, we need a free variable:

$$\mathbf{x} = \begin{pmatrix} 1 - 2x_2 \\ x_2 \end{pmatrix}$$





Systems of three linear equations

- Given a system of three linear equations in three unknowns, you apply the operations we described to eliminate variables in subsequent equations

$$\begin{array}{lll} 2x_1 + 3x_2 + x_3 = -0.7 & 2x_1 + 3x_2 + x_3 = -0.7 & 2x_1 + 3x_2 + x_3 = -0.7 \\ x_1 + 2x_2 + 4x_3 = 0.1 & \sim 0.5x_2 + 3.5x_3 = 0.45 & \sim 0.5x_2 + 3.5x_3 = 0.45 \\ -3x_1 + 4x_2 + 2x_3 = -2.5 & 8.5x_2 + 3.5x_3 = -3.55 & -56x_3 = -11.2 \\ \\ -56x_3 = -11.2 & 0.5x_2 + 3.5(0.2) = 0.45 & 2x_1 + 3(-0.5) + (0.2) = -0.7 \\ x_3 = 0.2 & x_2 = -0.5 & x_1 = 0.3 \end{array}$$





For higher-order systems of linear equations

- Given a system of n linear equations in n unknowns, we would proceed as follows:

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n = b_2$$

$$a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + \cdots + a_{3,n}x_n = b_3$$

$$\vdots$$

$$a_{n,1}x_1 + a_{n,2}x_2 + a_{n,3}x_3 + \cdots + a_{n,n}x_n = b_n$$





Moving forward

- Working with equations is tedious,
so moving forward, we will introduce the augmented
matrix representation of a system of linear equations
and define an algorithm on that matrix to find a solution





Summary

- Following this topic, you now
 - Have recalled some of the rules you used to solve a system of linear equations in secondary school
 - Know that there may be no solutions, a unique solution or infinitely many solutions
 - Know that a homogenous linear equation has a right-hand zero
 - Know that there are three operations you can perform on equations that leave the solution unchanged
 - Have reviewed solving systems of two and three linear equations with as many unknowns





References

- [1] https://en.wikipedia.org/wiki/System_of_linear_equations
- [2] https://en.wikipedia.org/wiki/Nonlinear_system





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Colophon

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